

ON A GENERALIZATION OF CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. A motivation comes from *M. Ismail and et al.: A generalization of starlike functions, Complex Variables Theory Appl., 14 (1990), 77–84* to study a generalization of close-to-convex functions by means of a q -analog of a difference operator acting on analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We use the terminology *q -close-to-convex functions* for the q -analog of close-to-convex functions. The q -theory has wide applications in special functions and quantum physics which makes the study interesting and pertinent in this field. In this paper, we obtain some interesting results concerning conditions on the coefficients of power series of functions analytic in the unit disk which ensure that they generate functions in the q -close-to-convex family. As a result we find certain dilogarithm functions that are contained in this family. Secondly, we also study the famous Bieberbach conjecture problem on coefficients of analytic q -close-to-convex functions. This produces several power series of analytic functions convergent to basic hypergeometric functions.

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1. INTRODUCTION

Denote by \mathcal{A} , the class of functions $f(z)$, normalized by $f(0) = 0 = f'(0) - 1$, that are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In other words, the functions $f(z)$ in \mathcal{A} have the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

We denote by \mathcal{S} , the class of univalent (i.e. analytic and one-one) functions in \mathbb{D} . Denote by \mathcal{S}^* , the subclass consisting of functions $f(z)$ in \mathcal{S} that are starlike with respect to the origin, i.e. $tw \in f(\mathbb{D})$ whenever $t \in [0, 1]$ and $w \in f(\mathbb{D})$. Analytically, it is well-known that $f(z) \in \mathcal{S}^*$ if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D};$$

and a function $f(z) \in \mathcal{A}$ is said to be *close-to-convex* if there exists $g(z) \in \mathcal{S}^*$ such that

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0, \quad z \in \mathbb{D}.$$

Then we say that $f \in \mathcal{K}$ with the function g . The class of close-to-convex functions defined in the unit disk is denoted by \mathcal{K} . One can easily verify the fact that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$ (see for

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instance [5]). For several interesting geometric properties of these classes, one can refer to the standard books [9, 17].

A q -analog of the class of starlike functions was first introduced in [12] by means of the q -difference operator $(D_q f)(z)$ acting on functions $f(z) \in \mathcal{A}$ defined by

$$(1.1) \quad (D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathbb{D} \setminus \{0\}, \quad (D_q f)(0) = f'(0),$$

where $q \in (0, 1)$. Note that the q -difference operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [2, 6, 7, 13, 24]). One can clearly see that $(D_q f)(z) \rightarrow f'(z)$ as $q \rightarrow 1^-$. This difference operator helps us to generalize the class of starlike functions \mathcal{S}^* analytically. We denote by \mathcal{S}_q^* , the class of functions in this generalized family. For the sake of convenience, we also call functions in \mathcal{S}_q^* the q -starlike functions. This is defined as follows:

Definition 1.2. A function $f \in \mathcal{A}$ is said to belong to the class \mathcal{S}_q^* if

$$\left| \frac{z}{f(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Clearly, when $q \rightarrow 1^-$, the class \mathcal{S}_q^* will coincide with \mathcal{S}^* .

As \mathcal{S}_q^* generalizes \mathcal{S}^* in the above manner, a similar form of q -analog of close-to-convex functions was expected and it is defined in the following form (see [20]).

Definition 1.3. A function $f \in \mathcal{A}$ is said to belong to the class \mathcal{K}_q if there exists $g \in \mathcal{S}^*$ such that

$$\left| \frac{z}{g(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Then we say that $f \in \mathcal{K}_q$ with the function g .

In [20], the authors have investigated some basic properties of functions that are in \mathcal{K}_q . Some of these results are also recalled in this paper in order to exhibit their interesting consequences. As $(D_q f)(z) \rightarrow f'(z)$, as $q \rightarrow 1^-$, we observe in the limiting sense that the closed disk $|w - (1-q)^{-1}| \leq (1-q)^{-1}$ becomes the right half-plane $\operatorname{Re}(zf'(z)/g(z)) > 0$ and hence the class \mathcal{K}_q clearly reduces to \mathcal{K} . In this paper, we refer to the functions in the class \mathcal{K}_q the q -close-to-convex functions. For the sake of convenience, we use the notation \mathcal{S}_q^* instead of the notation PS_q used in [12] and \mathcal{K}_q instead of PK_q used in [20]. It is easy to see that $\mathcal{S}_q^* \subset \mathcal{K}_q$ for all $q \in (0, 1)$. Clearly, one can easily see from the above discussion that

$$\bigcap_{0 < q < 1} \mathcal{K}_q \subset \mathcal{K} \subset \mathcal{S}.$$

Our main aim in this paper is to consider the following two ideas.

The first idea has its genesis in the work of Frideman [8]. He proved that there are only nine functions in the class \mathcal{S} whose coefficients are rational integers. They are

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

It is easy to see that these functions map the unit disk \mathbb{D} onto starlike domains. Using the idea of MacGregor [14], we derive some sufficient conditions for functions to be in \mathcal{K}_q whose coefficients are connected with certain monotone properties. These sufficient conditions help us

to examine functions of dilogarithm types [13, 25] which are in the \mathcal{K}_q family. Certain special functions, which are in the starlike and close-to-convex family, have been well-investigated in [11, 15, 16, 18, 19, 22, 23].

The second idea deals with the famous Bieberbach conjecture problem in analytic univalent function theory [4, 5]. A necessary and sufficient condition for a function $f(z)$ to be in \mathcal{S}_q^* is obtained in [12] by means of an integral representation of the function $zf'(z)/f(z)$ which yields the maximum moduli of coefficients of f . Using this condition, the Bieberbach conjecture problem for q -starlike functions has been solved in the following form.

Theorem A. [12, Theorem 1.18] *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class \mathcal{S}_q^* , then $|a_n| \leq c_n$ with equality holds for all n if and only if $f(z)$ is a rotation of*

$$k_q(z) := z \exp \left[\sum_{n=1}^{\infty} \frac{-2 \ln q}{1 - q^n} z^n \right] = z + \sum_{n=2}^{\infty} c_n z^n, \quad z \in \mathbb{D}.$$

Note that the function $k_q(z)$ plays a role of the Koebe function $k(z)$. By differentiating once the above expression for $k_q(z)$ and comparing the coefficients of z^{n-1} in both sides, we get the recurrence relation in c_n :

$$c_2 = \frac{-2 \ln q}{1 - q} \quad \text{and} \quad (n-1)c_n = \frac{-2 \ln q}{1 - q^{n-1}}(n-1) + \sum_{k=2}^{n-1} \frac{-2 \ln q}{1 - q^{k-1}} c_{n+1-k}(k-1), \quad n \geq 3.$$

It can be easily verified that Theorem A turns into the famous conjecture of Bieberbach (known as Bieberbach-de Branges Theorem) for the class \mathcal{S}^* , if $q \rightarrow 1^-$. Comparing with the Bieberbach-de Branges theorem for close-to-convex functions, one would expect that Theorem A also holds true for q -close-to-convex functions. However, this problem remains an open problem. Indeed, in this manuscript, we obtain an optimal coefficient bound for q -close-to-convex functions leading to the Bieberbach-de Branges theorem for close-to-convex functions, when $q \rightarrow 1^-$. Finally, for a special attention, we collect few consequences of the Bieberbach-de Branges theorem for the class \mathcal{K}_q with respect to the nine starlike functions considered above.

2. PROPERTIES FOR $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ TO BE IN \mathcal{K}_q

In this section, we mainly concentrate on problems in situations where the co-efficients A_n of functions $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ in \mathcal{K}_q are real, non-negative and connected with certain monotone properties. Similar investigations for the class of close-to-convex functions are studied in [1, 14] (see references there in for initial contributions of Fejér and Szegő in this direction).

We obtain several sufficient conditions for the representation $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ to be in \mathcal{K}_q . Rewriting this representation, we get

$$(2.1) \quad f(z) = \sum_{n=0}^{\infty} A_n z^n \quad (A_0 = 0, A_1 = 1).$$

If $f(z)$ is of the form (2.1), then a simple computation yields

$$(2.2) \quad (D_q f)(z) = 1 + \sum_{n=2}^{\infty} \frac{A_n(1-q^n)}{1-q} z^{n-1}$$

for all $z \in \mathbb{D}$. With this, we now collect a number of sufficient conditions for functions to be in \mathcal{K}_q .

Lemma 2.1. [20, Lemma 1.1(1)] *Let $f(z)$ be of the form (2.1) and $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$, with $B_n = A_n(1-q^n)/(1-q)$. Then $f(z) \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.*

As a consequence of Lemma 2.1, we have

Theorem 2.2. *Let $\{A_n\}$ be a sequence of real numbers such that $B_n = A_n(1-q^n)/(1-q)$ for all $n \geq 1$. Suppose that*

$$1 \geq B_2 \geq B_3 \geq \cdots \geq B_n \geq \cdots \geq 0 \quad \text{or} \quad 1 \leq B_2 \leq B_3 \leq \cdots \leq B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.

Proof. We know that

$$\sum_{n=1}^{\infty} |B_{n+1} - B_n| = \lim_{k \rightarrow \infty} \sum_{n=1}^k |B_{n+1} - B_n|.$$

If $1 \geq B_2 \geq B_3 \geq \cdots \geq B_n \geq \cdots \geq 0$, we see that

$$\lim_{k \rightarrow \infty} \sum_{n=1}^k |B_{n+1} - B_n| = \lim_{k \rightarrow \infty} (B_1 - B_{k+1}) \leq B_1 = 1.$$

Similarly, if $1 \leq B_2 \leq B_3 \leq \cdots \leq B_n \leq \cdots \leq 2$ then we get $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq 1$. Thus, by Lemma 2.1, we prove the assertion of our theorem. \square

Remark. If we choose the limit $q \rightarrow 1^-$ in Theorem 2.2, one can obtain the results of Alexander [1] and MacGregor [14].

Example. The quantum dilogarithm function is defined by

$$Li_2(z; q) = \sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}, \quad |z| < 1, \quad 0 < q < 1.$$

Note that this function is studied by Kirillov [13] (see also [25, p.28]) and is a q -deformation of the ordinary *dilogarithm function* [13] defined by $Li_2(z) = \sum_{n=1}^{\infty} (z^n/n^2)$, $|z| < 1$, in the sense that

$$\lim_{\epsilon \rightarrow 0} \epsilon Li_2(z; e^{-\epsilon}) = Li_2(z).$$

By Theorem 2.2, one can ascertain that the function $(1-q)Li_2(z; q) \in \mathcal{K}_q$.

Theorem 2.3. *Let f be defined by (2.1) and suppose that*

$$\sum_{n=1}^{\infty} |B_n - B_{n-1}| \leq 1, \quad B_n = \frac{A_{n+1}(1-q^{n+1})}{1-q} - \frac{A_n(1-q^n)}{1-q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)^2$.

Proof. Starting with $|B_n|$, we see that

$$|B_n| = \left| \sum_{k=1}^n (B_k - B_{k-1}) + 1 \right| \leq \sum_{k=1}^n |B_k - B_{k-1}| + 1 \leq 2.$$

Hence, for all $n \geq 2$, we have

$$\left| \frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q} \right| \leq 2.$$

Now, by using the repeated triangle inequality, we see that

$$\begin{aligned} \left| \frac{A_n(1-q^n)}{1-q} \right| &= \left| \frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q} + \frac{A_{n-1}(1-q^{n-1})}{1-q} - \frac{A_{n-2}(1-q^{n-2})}{1-q} \right. \\ &\quad \left. + \cdots + \frac{A_2(1-q^2)}{1-q} - 1 + 1 \right| \\ &\leq 2(n-1) + 1 = 2n-1 \end{aligned}$$

and so $|A_n| \leq (2n-1)/(1+q+\cdots+q^{n-1})$. By applying the root test, one can see that the radius of convergence of $\sum_{n=0}^{\infty} A_n z^n$ is not less than unity. Therefore, $f \in \mathcal{A}$.

Since f is of the form (2.1), we compute by using (2.2) that

$$\begin{aligned} (1-z)^2(D_q f)(z) &= 1 + \frac{A_2(1-q^2)}{1-q}z - 2z \\ &\quad + \sum_{n=3}^{\infty} \left[\frac{A_n(1-q^n)}{1-q} - \frac{2A_{n-1}(1-q^{n-1})}{1-q} + \frac{A_{n-2}(1-q^{n-2})}{1-q} \right] z^{n-1}. \end{aligned}$$

By the definition of B_n as given in the hypothesis, we have

$$(1-z)^2(D_q f)(z) = 1 + (B_1 - 1)z + \sum_{n=3}^{\infty} (B_{n-1} - B_{n-2})z^{n-1}.$$

Hence,

$$\frac{1}{1-q} - \left| (1-z)^2(D_q f)(z) - \frac{1}{1-q} \right| \geq 1 - |B_1 - 1| - \sum_{n=3}^{\infty} |B_{n-1} - B_{n-2}| \geq 0,$$

if $\sum_{n=2}^{\infty} |B_{n-1} - B_{n-2}| \leq 1$. This proves the assertion of our theorem. \square

By Theorem 2.3, we immediately have the following result which generalizes couple of results of MacGregor (see [14, Theorems 3 and 5]).

Theorem 2.4. *Let $\{A_n\}$ be a sequence of real numbers such that*

$$A_0 = 0, \quad A_1 = 1 \quad \text{and} \quad B_n = \frac{A_{n+1}(1-q^{n+1})}{1-q} - \frac{A_n(1-q^n)}{1-q}.$$

Suppose that

$$1 \geq B_1 \geq B_2 \geq \cdots \geq B_n \geq \cdots \geq 0 \quad \text{or} \quad 1 \leq B_1 \leq B_2 \leq \cdots \leq B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z)^2$.

Theorem 2.5. *Let f be defined by $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1}$ and suppose that*

$$\sum_{n=1}^{\infty} |B_{2n-1} - B_{2n+1}| \leq 1, \quad B_n = \frac{A_n(1 - q^n)}{1 - q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z^2)$.

Proof. First of all we shall prove that $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1} \in \mathcal{A}$. For this, we estimate

$$|B_{2n+1}| = \left| \sum_{k=1}^n (B_{2k-1} - B_{2k+1}) - 1 \right| \leq 2$$

so that $|A_n| \leq 2/(1 + q + \cdots + q^{n-1})$. By applying the root test, one can see that the radius of convergence of the series expansion of $f(z)$ is not less than unity. Therefore, $f \in \mathcal{A}$.

Since $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1}$, by (1.1) we get

$$(1 - z^2)(D_q f)(z) = 1 - \sum_{n=1}^{\infty} \left[\frac{A_{2n-1}(1 - q^{2n-1})}{1 - q} - \frac{A_{2n+1}(1 - q^{2n+1})}{1 - q} \right] z^{2n}.$$

Note that $B_n = A_n(1 - q^n)/(1 - q)$. So, we have

$$\frac{1}{1 - q} - \left| (1 - z^2)(D_q f)(z) - \frac{1}{1 - q} \right| \geq 1 - \sum_{n=1}^{\infty} |B_{2n-1} - B_{2n+1}| \geq 0,$$

whenever $\sum_{n=1}^{\infty} |B_{2n-1} - B_{2n+1}| \leq 1$. This proves the conclusion of our theorem. \square

By Theorem 2.5, we immediately have the following result which generalizes a result of MacGregor (see [14, Theorem 2]).

Theorem 2.6. *Let $\{A_n\}$ be a sequence of real numbers such that $B_n = A_n(1 - q^n)/(1 - q)$ for all $n \geq 1$. Suppose that*

$$1 \geq B_3 \geq B_5 \geq \cdots \geq B_{2n-1} \geq \cdots \geq 0 \quad \text{or} \quad 1 \leq B_3 \leq B_5 \leq \cdots \leq B_{2n-1} \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1} \in \mathcal{K}_q$ with $g(z) = z/(1 - z^2)$.

Lemma 2.7. [20, Lemma 1.1(4)] *Let f be defined by (2.1) and suppose that*

$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1, \quad B_n = \frac{A_n(1 - q^n)}{(1 - q)}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z^2)$.

Lemma 2.7 leads the following sufficient conditions for functions to be in \mathcal{K}_q .

Theorem 2.8. *Let $\{A_n\}$ be a sequence of real numbers such that*

$$A_1 = 1 \quad \text{and} \quad B_n = \frac{A_n(1 - q^n)}{1 - q}$$

for all $n \geq 1$. Suppose that

$$1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0 \quad \text{or} \quad 1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1 - z^2)$.

Proof. We know that

$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| = \lim_{k \rightarrow \infty} \sum_{n=2}^k |B_n - B_{n-2}|.$$

If $1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0$, we see that

$$\lim_{k \rightarrow \infty} \sum_{n=2}^k |B_n - B_{n-2}| = \lim_{k \rightarrow \infty} (1 - B_{k-1} - B_k) \leq 1 + 0 = 1.$$

Similarly, if $1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2$ then $\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1$. Thus, by Theorem 2.7, we complete the proof. \square

As a consequence of Theorem 2.8, one can obtain the following new criteria for functions to be in the close-to-convex family.

Theorem 2.9. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and $b_n = na_n$ for all $n \geq 1$. Suppose that*

$$1 \geq b_1 + b_2 \geq \cdots \geq b_{n-1} + b_n \geq \cdots \geq 0 \quad \text{or} \quad 1 \leq b_1 + b_2 \leq \cdots \leq b_{n-1} + b_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is close-to-convex with $g(z) = z/(1 - z^2)$.

Lemma 2.10. [20, Lemma 1.1(2)] *Let f be defined by (2.1) and suppose that*

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1, \quad B_n = \frac{A_n(1 - q^n)}{1 - q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z + z^2)$.

Lemma 2.10 yields the following sufficient condition.

Theorem 2.11. *Let $\{A_n\}$ be a sequence of real numbers such that*

$$A_1 = 1 \quad \text{and} \quad B_n = \frac{A_n(1 - q^n)}{1 - q}$$

for all $n \geq 1$. Suppose that

$$0 \geq B_2 - B_1 \geq B_3 \geq B_2 + B_4 \geq B_2 + B_3 + B_5 \geq \cdots \geq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \geq -1$$

or

$$0 \leq B_2 - B_1 \leq B_3 \leq B_2 + B_4 \leq B_2 + B_3 + B_5 \leq \cdots \leq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \leq 1$$

holds. Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1 - z + z^2)$.

Proof. We know that

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| = \lim_{k \rightarrow \infty} \sum_{n=1}^k |B_{n-1} - B_n + B_{n+1}|.$$

If

$$0 \geq B_2 - B_1 \geq B_3 \geq B_2 + B_4 \geq B_2 + B_3 + B_5 \geq \cdots \geq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \geq -1,$$

we see that

$$\lim_{k \rightarrow \infty} \sum_{n=1}^k |B_{n-1} - B_n + B_{n+1}| = \lim_{k \rightarrow \infty} -(B_2 + B_3 + B_4 + \cdots + B_{k-1} + B_{k+1}) \leq 1.$$

Similarly, if

$$0 \leq B_2 - B_1 \leq B_3 \leq B_2 + B_4 \leq B_2 + B_3 + B_5 \leq \cdots \leq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \leq 1$$

then one can obtain $\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1$. Thus, by Theorem 2.10, we complete the proof. \square

As a result of Theorem 2.11, one can obtain the following new criteria for functions to be in the close-to-convex family.

Theorem 2.12. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and $b_n = na_n$, for all $n \geq 1$. Suppose that*

$$0 \geq b_2 - b_1 \geq b_3 \geq b_2 + b_4 \geq b_2 + b_3 + b_5 \geq \cdots \geq b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \geq -1$$

or

$$0 \leq b_2 - b_1 \leq b_3 \leq b_2 + b_4 \leq b_2 + b_3 + b_5 \leq \cdots \leq b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \leq 1.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the close-to-convex family with respect to $g(z) = z/(1 - z + z^2)$.

3. THE BIEBERBACH-DE BRANGES THEOREM FOR \mathcal{K}_q

A necessary and sufficient condition for functions to be in \mathcal{S}_q^* was obtained in [12, Theorem 1.5] in the following form: a function $f \in \mathcal{S}_q^*$ if and only if $|f(qz)/f(z)| \leq 1$ for all $z \in \mathbb{D}$.

A similar characterization for functions in \mathcal{K}_q is

Lemma 3.1. *A function $f \in \mathcal{K}_q$ if and only if there exists $g \in \mathcal{S}^*$ such that*

$$\frac{|g(z) + f(qz) - f(z)|}{|g(z)|} \leq 1 \quad \text{for all } z \in \mathbb{D}.$$

Proof. The proof follows immediately after making the substitution for the expression of the q -difference operator $(D_q f)(z)$ in Definition 1.3. \square

In this section, Lemma 3.1 will act as one of the crucial results to estimate coefficient bounds for series representation of functions in the class \mathcal{K}_q , i.e. in other words, we analyze the Bieberbach-de Branges theorem for the class of q -close-to-convex functions. The Bieberbach conjecture for close-to-convex functions is proved by Reade [21] (see also [9] for more details). It states that if $f \in \mathcal{K}$, then $|a_n| \leq n$ for all $n \geq 2$.

We now proceed to state and prove the Bieberbach-de Branges Theorem for functions in the q -close-to-convex family.

Theorem 3.2 (Bieberbach-de Branges Theorem for \mathcal{K}_q). *If $f \in \mathcal{K}_q$, then*

$$|a_n| \leq \frac{1-q}{1-q^n} \left[n + \frac{n(n-1)}{2}(1+q) \right] \quad \text{for all } n \geq 2.$$

Proof. Since $f \in \mathcal{K}_q$, by Lemma 3.1 there exists $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that

$$(3.1) \quad g(z) + f(qz) - f(z) = w(z)g(z).$$

Clearly $w(0) = q$. By assuming $a_1 = 1 = b_1$, we then have

$$\sum_{n=1}^{\infty} (b_n + a_n q^n - a_n) z^n = \sum_{n=1}^{\infty} q b_n z^n + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} w_{n-k} b_k \right) z^n.$$

Equating the coefficients of z^n , for $n \geq 2$, we obtain

$$a_n(q^n - 1) = b_n(q - 1) + \sum_{k=1}^{n-1} w_{n-k} b_k.$$

From the classical result [3], one can verify that $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$. Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$, we get

$$|a_n| \leq \frac{1-q}{1-q^n} \left[n + (1+q) \sum_{k=1}^{n-1} k \right] \quad \text{for all } n \geq 2.$$

This proves the conclusion of our theorem. \square

Remark. When $q \rightarrow 1^-$, certainly Theorem 3.2 yields the Bieberbach conjecture problem for close-to-convex functions.

It is easy to see, by the usual ratio test, that the series

$$(3.2) \quad z + \sum_{n=2}^{\infty} \frac{1-q}{1-q^n} \left[n + \frac{n(n-1)}{2} (1+q) \right] z^n$$

converges for $|z| < 1$. Indeed, we can ascertain by using the convergence factor for the series $\sum_{n=1}^{\infty} z^n / (1 - q^n)$ (see [24, 3.2.2.1]) that the series given by (3.2) converges to the function

$$\frac{1+q}{2} z^2 \frac{d^2 \Psi(q; z)}{dz^2} + z \frac{d \Psi(q; z)}{dz},$$

where $\Psi(q; z) := z \Phi[q, q; q^2; q, z]$ represents its Heine hypergeometric function. Note that the q -hypergeometric series was developed by Heine [10] as a generalization of the well-known Gauss hypergeometric series:

$$\Phi[a, b; c; q, z] = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} z^n, \quad |q| < 1, \quad 1 \neq cq^n, \quad |z| < 1,$$

where the q -shifted factorial $(a; q)_n$ is defined by

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}) \quad \text{and} \quad (a; q)_0 = 1.$$

This is also known as the basic hypergeometric series and its convergence function is known as the basic hypergeometric function. We refer one of the standard books [24] for the notation of the basic hypergeometric function. For history of q -series related calculus and their applications, we suggest readers to refer [6].

Due to Frideman's result, we now study the special cases of Theorem 3.2 with respect to the nine functions having integer coefficients. However, in this situation, it is enough to consider the identity function and four other functions which contain factors $1 - z$ instead of $1 \pm z$ in the denominator. In particular, Theorem 3.2 reduces to the following corollaries. Note

that we provide proofs of last two consequences as they involve variations in the exponents, whereas the first three consequences follow directly after making precise substitution for the starlike functions $g(z)$.

Corollary 3.3. *If $f \in \mathcal{K}_q$ with the Koebe function $g(z) = z/(1 - z)^2$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1 - q}{1 - q^n} \left[n + (1 + q) \frac{n(n - 1)}{2} \right].$$

If $f \in \mathcal{K}$ with $g(z) = z$, then for all $n \geq 2$ it is well-known that $|a_n| \leq 2/n$. As a generalization, we have the following:

Corollary 3.4. *If $f \in \mathcal{K}_q$ with $g(z) = z$, then for all $n \geq 2$ we have $|a_n| \leq (1 - q^2)/(1 - q^n)$.*

Here we note that the series $z + \sum_{n=2}^{\infty} (1 - q^2)/(1 - q^n) z^n$ converges to the Heine hypergeometric function $(z + qz)\Phi[q, q; q^2; q, z] - qz = z + z^2\Phi[q^2, q^2; q^3; q^2, z]$ and it follows from [24, 3.2.2, pp. 91].

If $f \in \mathcal{K}$ with $g(z) = z/(1 - z)$, then for all $n \geq 2$ it is known that $|a_n| \leq (2n - 1)/n$. We find the following analogous result:

Corollary 3.5. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z)$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1 - q}{1 - q^n} [n + q(n - 1)].$$

One can similarly verify that the series $z + \sum_{n=2}^{\infty} \frac{1 - q}{1 - q^n} [n + q(n - 1)] z^n$ converges to the function $z(1 + q) \frac{d}{dz} \Psi(q; z) - q\Psi(q; z)$, where $\Psi(q; z) := z\Phi[q, q; q^2; q, z]$ represents its Heine hypergeometric function.

If $f \in \mathcal{K}$ with $g(z) = z/(1 - z^2)$, then for all $m \geq 1$ it is known that

$$|a_n| \leq \begin{cases} 1, & \text{if } n = 2m - 1; \\ 1, & \text{if } n = 2m. \end{cases}.$$

As a generalization, we now state the following corollary along with an outline of its proof:

Corollary 3.6. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \frac{1 - q}{1 - q^n} \left(\frac{n}{2}(1 + q) + \frac{1}{2}(1 - q) \right), & \text{if } n = 2m - 1; \\ \left(\frac{1 - q^2}{1 - q^n} \right) \frac{n}{2}, & \text{if } n = 2m. \end{cases}.$$

Proof. Since $g(z) = \frac{z}{1 - z^2} = \sum_{n=1}^{\infty} z^{2n-1}$, by (3.1) we get

$$\sum_{n=1}^{\infty} (q^n - 1) a_n z^n = (q - 1) \sum_{n=1}^{\infty} z^{2n-1} + \left(\sum_{n=1}^{\infty} z^{2n-1} \right) \left(\sum_{n=1}^{\infty} w_n z^n \right).$$

This is equivalent to

$$(3.3) \quad \sum_{n=1}^{\infty} (q^n - 1) a_n z^n = (q - 1) \sum_{n=1}^{\infty} z^{2n-1} + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} w_{2k} \right) z^{2n-1} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n w_{2k-1} \right) z^{2n}.$$

In order to prove the required optimal bound for $|a_n|$, in this situation, it is appropriate to compare the coefficients of z^{2n-1} and z^{2n} separately.

In (3.3), first we compare the coefficients of z^{2n-1} , for $n \geq 2$, we get

$$(q^{2n-1} - 1)a_{2n-1} = (q - 1) + \sum_{k=1}^{n-1} w_{2k}.$$

Since $|w_k| \leq (1 - q^2)$ for all $k \geq 1$ and $q \in (0, 1)$, we have

$$|a_{2n-1}| \leq \frac{1 - q}{(1 - q^{2n-1})} (-q + (1 + q)n).$$

Secondly, by comparing the coefficients of z^{2n} , for $n \geq 1$, we obtain

$$(q^{2n} - 1)a_{2n} = \sum_{k=1}^n w_{2k-1},$$

and similarly we get the bound

$$|a_{2n}| \leq \frac{1 - q}{(1 - q^{2n})} (1 + q)n.$$

Thus, we prove the required optimal bound for $|a_n|$. □

If $f \in \mathcal{K}$ with $g(z) = z/(1 - z + z^2)$, then for all $n \geq 2$ it is known that

$$|a_n| \leq \begin{cases} \frac{4n+1}{3n}, & \text{if } n = 3m - 1; \\ \frac{4}{3}, & \text{if } n = 3m; \\ \frac{4n-1}{3n}, & \text{if } n = 3m + 1. \end{cases}.$$

As a generalization, we have the following:

Corollary 3.7. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z + z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \frac{1 - q}{1 - q^n} \left(\frac{1}{3}(2 - q) + \frac{2n}{3}(1 + q) \right), & \text{if } n = 3m - 1; \\ \frac{1 - q^2}{1 - q^n} \frac{2n}{3}, & \text{if } n = 3m; \\ \frac{1 - q}{1 - q^n} \left(\frac{2n}{3}(1 + q) + \frac{1}{3}(1 - 2q) \right), & \text{if } n = 3m + 1. \end{cases}.$$

Proof. By rewriting the function $g(z) = z/(1 - z + z^2)$, we obtain

$$g(z) = \frac{z(1 + z)}{1 + z^3} = \sum_{n=1}^{\infty} (-1)^{n-1} z^{3n-2} + \sum_{n=1}^{\infty} (-1)^{n-1} z^{3n-1}.$$

Then simplifying the relation (3.1), we get

$$\begin{aligned}
& \sum_{n=1}^{\infty} (q^n - 1) a_n z^n \\
&= (q - 1) \left(\sum_{n=1}^{\infty} (-1)^{n-1} z^{3n-2} + \sum_{n=1}^{\infty} (-1)^{n-1} z^{3n-1} \right) \\
&+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k} w_{3k-2} \right) z^{3n-1} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k} w_{3k-1} \right) z^{3n} \\
&+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k} w_{3k} \right) z^{3n+1} + \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} (-1)^{n-k} w_{3k} \right) z^{3n-1} \\
(3.4) \quad &+ \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k} w_{3k-2} \right) z^{3n} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^{n-k-1} w_{3k-1} \right) z^{3n+1}.
\end{aligned}$$

First equating the coefficients of z^{3n-1} , for $n \geq 2$, in (3.4), we get

$$(q^{3n-1} - 1) a_{3n-1} = (-1)^{n-k} (q - 1) + \sum_{k=1}^n (-1)^{n-k} w_{3k-2} + \sum_{k=1}^n (-1)^{n-k} w_{3k}.$$

Since $|w_k| \leq (1 - q^2)$ for all $k \geq 1$ and $q \in (0, 1)$, we have

$$|a_{3n-1}| \leq \frac{1 - q}{(1 - q^{3n-1})} (-q + 2(1 + q)n).$$

Next, for all $n \geq 1$, we compare the coefficients of z^{3n} and z^{3n+1} in (3.4), we respectively obtain the coefficient bounds

$$|a_{3n}| \leq \frac{2(1 - q)}{(1 - q^{3n})} (1 + q)n \quad \text{and} \quad |a_{3n+1}| \leq \frac{(1 - q)}{(1 - q^{3n+1})} (1 + 2(1 + q)n).$$

Thus, the assertion of our corollary follows. \square

Remark. By making use of Lemma [12, Theorem 1.5], one can also obtain the Bieberbach-de Branges theorem for \mathcal{S}_q^* as follows. This also yields the Bieberbach-de Branges theorem for \mathcal{S}^* , in particular. However, it defers from Theorem A.

4. APPENDIX

In this section, we verify that a similar technique used in the previous section yields a form of the Bieberbach-de Branges theorem for \mathcal{S}_q^* . This leads to the coefficient problem of Bieberbach-de Branges (different from Theorem A!) for the class \mathcal{S}^* , when $q \rightarrow 1^-$, as well.

Theorem 4.1 (The Bieberbach-de Branges Theorem for \mathcal{S}_q^*). *If $f \in \mathcal{S}_q^*$, then for all $n \geq 2$ we have*

$$(4.1) \quad |a_n| \leq \left(\frac{1 - q^2}{q - q^n} \right) \prod_{k=2}^{n-1} \left(1 + \frac{1 - q^2}{q - q^k} \right).$$

Proof. We know that $f \in \mathcal{S}_q^*$ if and only if

$$|f(qz)/f(z)| \leq 1 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $w : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that

$$\frac{f(qz)}{f(z)} = w(z), \quad \text{i.e. } f(qz) = w(z)f(z) \text{ for all } z \in \mathbb{D}.$$

Clearly, $w(0) = q$. In terms of series expansion, we get (with $a_1 = 1$ and $w_0 = q$)

$$\sum_{n=1}^{\infty} a_n q^n z^n = \left(\sum_{n=0}^{\infty} w_n z^n \right) \left(\sum_{n=1}^{\infty} a_n z^n \right) =: \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n := \sum_{k=1}^n w_{n-k} a_k = q a_n + \sum_{k=1}^{n-1} w_{n-k} a_k$. Comparing the coefficients of z^n ($n \geq 2$), we get

$$a_n(q^n - q) = \sum_{k=1}^{n-1} w_{n-k} a_k, \quad \text{for } n \geq 2.$$

Since $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$, we see that

$$|a_n| \leq \frac{1 - q^2}{q - q^n} \sum_{k=1}^{n-1} |a_k| \quad \text{for each } n \geq 2.$$

Thus for $n = 2$, one has $|a_2| \leq (1 - q^2)/(q - q^2)$, and for $n \geq 3$, we apply a similar technique to estimate $|a_{n-1}|$ and get

$$|a_n| \leq \frac{1 - q^2}{q - q^n} \left(1 + \frac{1 - q^2}{q - q^{n-1}} \right) \sum_{k=1}^{n-2} |a_k|.$$

Iteratively, we conclude that

$$|a_n| \leq \frac{1 - q^2}{q - q^n} \left(1 + \frac{1 - q^2}{q - q^{n-1}} \right) \left(1 + \frac{1 - q^2}{q - q^{n-2}} \right) \cdots \left(1 + \frac{1 - q^2}{q - q^2} \right)$$

for all $n \geq 3$. This completes the proof. \square

Remark. One can easily verify that the right hand side of (4.1) approaches n as $q \rightarrow 1^-$, which will lead to the Bieberbach-de Branges theorem for starlike functions [5, Theorem 2.14].

We also find that the ratio test easily provides the convergence of the series $z + \sum_{n=2}^{\infty} A_n z^n$ in the sub-disk $|z| < q/(q + 1 - q^2)$, where

$$A_n = \left(\frac{1 - q^2}{q - q^n} \right) \prod_{k=1}^{n-2} \left(1 + \frac{1 - q^2}{q - q^{k+1}} \right).$$

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